**Note:** In this homework you may find the following inequality useful. When $x \in (0,1)$:

$$\exp\left(-\frac{x}{1-x}\right) \leq 1 - x \leq \exp(-x).$$

A nice special case is that if $x \in (0, 1/2)$ then $1 - x \geq e^{-2x}$. Feel free to use these without proof!

1. **(10 pt.) [Threshold for isolation]**

   Recall that $G_{n,p}$ refers to a random graph with $n$ vertices, where each of the $\binom{n}{2}$ possible edges is present independently with probability $p$.

   (a) **(2 pt.)** Suppose that $p = 1.01\frac{\ln n}{n}$. Show that $G_{n,p}$ has an isolated vertex with probability $o(1)$.

   (b) **(4 pt.)** Let $X_1, X_2, ..., X_n$ be 0/1 random variables that are *not* necessarily independent, and *not* necessarily identically distributed, and let $X = \sum_{i=1}^{n} X_i$. Prove that

   $$\mathbb{E}[X^2] = \sum_{i=1}^{n} \Pr[X_i = 1] \cdot \mathbb{E}[X \mid X_i = 1].$$

   (c) **(4 pt.)** Suppose that $p = 0.99\frac{\ln n}{n}$. Show that $G_{n,p}$ has an isolated vertex with probability $1 - o(1)$.
   [**HINT:** *Consider using part (b) – it might make the math simpler.*]

**SOLUTION:**

(a) Fix a single vertex in $G_{n,p}$. It is isolated iff none of the $(n-1)$ potential edges adjacent to it exist in $G_{n,p}$. By independence of each such edge, the probability this vertex is isolated is

$$\left(1 - 1.01\frac{\ln n}{n}\right)^{n-1} \leq \exp(-1.01(n-1)/n \cdot \ln n).$$

For large enough $n$, $1.01(n-1)/n \geq 1.005$. In this regime, by union bound, the probability any vertex is isolated is at most

$$n \cdot \exp(-1.005 \cdot \ln n) = n \cdot n^{-1.005} = n^{-0.005} = o(1).$$

(b) Using the distributive property and linearity of expectation, we can write,

$$\mathbb{E}[X^2] = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}[X_i X_j].$$

We simplify the right hand side of the desired expression:

$$\begin{aligned}
\sum_{i=1}^{n} \Pr[X_i = 1] \cdot \mathbb{E}[X \mid X_i = 1] &= \sum_{i=1}^{n} \Pr[X_i = 1] \sum_{j=1}^{n} \mathbb{E}[X_j \mid X_i = 1] \\
&= \sum_{i=1}^{n} \Pr[X_i = 1] \sum_{j=1}^{n} \frac{\mathbb{E}[X_j X_i]}{\Pr[X_i = 1]} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}[X_i X_j] = \mathbb{E}[X^2].
\end{aligned}$$

(c) Let the vertices be labeled $\{1, 2, ..., n\}$, and let $X_i$ be the indicator that vertex $i$ is isolated. Let $X = \sum_{i=1}^{n} X_i$. We want to show that $\Pr[X = 0] = o(1)$. Recall that

$$\Pr[X = 0] \le \frac{\mathrm{Var}[X]}{(\mathbb{E}[X])^2} = \frac{\mathbb{E}[X^2]}{(\mathbb{E}[X])^2} - 1.$$

As we saw in part (a), $\Pr[X_i = 1] = (1-p)^{n-1}$, and so $\mathbb{E}[X] = n(1-p)^{n-1}$.
To compute the variance, we'll use part (b). Conditioned on $i$ being isolated, the expected total number of isolated vertices is 1 (for $i$) plus the expected number of isolated vertices in the graph with $i$ removed. That graph is just a $G_{n-1,p}$, so it follows that

$$\mathbb{E}[X \mid X_i = 1] = 1 + (n-1)(1-p)^{n-2}.$$

Then by part (b),

$$\mathbb{E}[X^2] = n(1-p)^{n-1}(1 + (n-1)(1-p)^{n-2}) = \mathbb{E}[X](1 + (n-1)(1-p)^{n-2}).$$

It follows that

$$\begin{aligned}
\Pr[X = 0] &\le \frac{1 + (n-1)(1-p)^{n-2}}{n(1-p)^{n-1}} - 1 \\
&= \frac{1}{n(1-p)^{n-1}} + \frac{1 - \frac{1}{n}}{1-p} - 1 \\
&\le \frac{1}{n(1-p)^{n-1}} + \frac{p}{1-p}.
\end{aligned}$$

Now, with our choice of $p$, we have $\frac{p}{1-p} = O(\frac{\ln n}{n}) = o(1)$. Next,

$$\frac{1}{n(1-p)^{n-1}} \le \frac{1}{n(1-p)^{n}} \le \frac{1}{n \exp(-\frac{pn}{1-p})} = \frac{1}{n^{1 - 0.99\frac{1}{1-p}}}.$$

Since $p \to 0$ as $n \to \infty$, this is $o(1)$. To be more concrete, once $n \ge 10^5$ we have $p \le 0.001$, and so $1 - \frac{0.99}{1-p} \ge 0.009$. This means that for sufficiently large $n$, the above expression is at most $n^{-0.009} = o(1)$. Hence $\Pr[X = 0] = o(1)$ as desired.

2. **(6 pt.) [Echoing paths]**

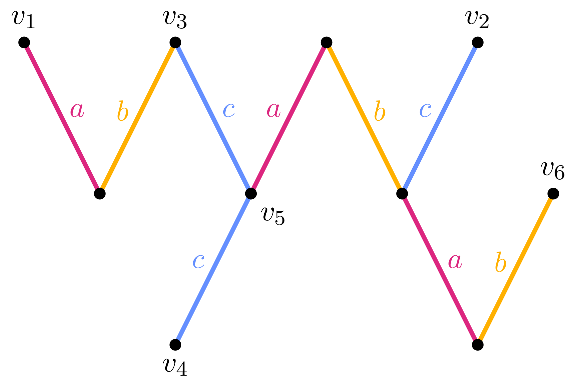


Figure 1: An edge coloring of a graph with some echoing paths.

An *edge coloring* of an (undirected) graph $G = (V, E)$ assigns exactly one color to each edge of the graph. We say that a colored path in the graph is *echoing* if the path has an even number of edges, and the second half of the path is colored identically to the first half of the path (i.e. the sequence of colors in the second half of the path is the same sequence as in the first half). For example, in Figure 1, the paths from $v_1$ and $v_2$, from $v_3$ to $v_4$, and from $v_5$ to $v_6$ are all echoing paths. Edges are colored and labeled $a, b$, or $c$ corresponding to their color.

Throughout this problem, by "path" we refer only to simple paths—i.e. paths that do not re-use any edges.

(a) **(4 pt.)** Prove that for any graph whose maximum degree is $d$, there exists a coloring using $10 \cdot d^2$ colors such that there are no echoing paths of length 4 (i.e. no echoing paths consisting of 4 distinct edges, like the path from $v_5$ to $v_6$ in Figure 1).
[**HINT:** *Lovasz Local Lemma!*]

(b) **(2 pt.)** Given the setup in the previous part, give an algorithm that will find such a coloring in expected time polynomial in the size of the graph, and justify the runtime.

(c) **(0 pt.) [This problem is optional.]** Prove that there is some constant $C$ such that for any graph whose maximum degree is $d$, there exists a coloring using $C \cdot d^2$ colors such that there are no echoing paths (of any length).

**SOLUTION:**

(a) We color each of the edges of the graphs independently at random with one of the $10d^2$ colors (each color equally likely for each edge). We will show using the LLL that with positive probability, this coloring has no symmetric paths of length 4.

For each path $p$ of length 4, let $B_p$ be the event that $p$ is a symmetric path in our coloring. $\Pr[B_p] = \frac{1}{(10d^2)} = \frac{1}{100d^4}$, since this is the probability that the third edge is the same as the first edge, and the fourth edge is the same as the first edge (using the independence of the edge colors). If two paths $p$ and $p'$ don't share an edge then $B_p$

and $B_{p'}$ they must be independent, since the coin flips that determine these events are disjoint.

For a fixed edge $e = (u, v)$, consider the number of paths $p$ of length 4 that use this edge. Any such path consists of some five vertices $v_1, v_2, v_3, v_4, v_5$, where $(v_i, v_{i+1})$ is an edge for each $1 \le i \le 4$, and $u = v_j$, $v = v_{j+1}$ for some $1 \le j \le 4$. For each choice of $j$, there are at most $d$ ways to extend the path (starting with just the one edge, $e$) to the left or right, and we have 3 such extensions to make, so there are at most $d^3$ such paths. As an example, if $j = 2$, then there are at most $d$ choices for $v_1$, and at most $d$ choices for $v_4$, and once $v_4$ is fixed, there are at most $d$ choices for $v_5$. This gives at most $d^3$ paths with $j = 2$. With 4 choices for $j$, it follows that there are at most $4d^3$ paths that include $e$.

Then, the number of paths that have a common edge with a fixed path $p$ is at most $16d^3$, since each such path must contain one of the four edges of $p$, and for each of those edges there are at most $4d^3$ possible paths. Hence, each event $B_p$ is independent from all but at most other $16d^3$ events. Since

$$\Pr[B_p] \cdot 16d^3 = \frac{16d^3}{100d^4} = \frac{1}{6.25d} < 1/4,$$

the conditions of LLL apply (Version 1 of Theorem 3 from Lecture 11), and so a coloring that avoids all bad events exists.

(b) We can directly apply the Moser-Tardos re-randomization algorithm. In this case the algorithm is as follows: The algorithm works by randomly assigning a random color to each edge, and then while there exists a symmetric path of length 4, choose one and randomly reassign colors to the edges in that path.

Initializing the colors takes time $O(n^2)$. We can naively iterate over each possible paths of length 4 and check if it is symmetric in time $O(n^5)$, so each iteration of the loop takes $O(n^5)$ time. The total number of bad events is also $O(n^5)$, since there are at most that many paths of length 4, and so Corollary 3 in Lecture 12 tells us the expected number of iterations needed is $O(n^5)$. Hence the overall expected runtime is $O(n^{10})$, which is polynomial in $n$.

(c) We proceed similarly to part (a), but instead we apply the Asymmetric LLL. Again we let $B_p$ be the event that a path $p$ of even length is symmetric. By the same argument as in part (a) if $p$ has length $2k$ then $\Pr[B_p] = \frac{1}{(Cd^2)^k} = \frac{1}{C^k d^{2k}}$.

Our goal will be to assign a value $r_p$ for each path $p$ such that the inequality for Theorem 5 in the lecture 11 notes is satisfied. To make things easier, we'll set $r_p = r_{p'}$ whenever $p$ and $p$ are paths of the same length. For ease, let $r_\ell$ be the value assigned to all paths of length $\ell$. We'll show that $r_\ell = \frac{1}{(cd)^\ell}$ works for an appropriate choice of $c$ which will depend on $C$. Eventually we will set $(c, C) = (4, 40)$ so these values can be kept in mind throughout the proof.

As in part (a), $B_p$ is mutually independent of all $B_{p'}$ such that $p'$ and $p$ do not share any edges. Given this, we bound the number of number of paths of length $\ell$ that are not independent from a fixed $B_p$ where $p$ is a path of length $2k$. By the same argument as in part (a) that there are at most $\ell d^{\ell-1} \le \ell d^\ell$ paths of length $\ell$ that use a fixed edge.

Hence, in total, the number of paths of length $\ell$ that share an edge with $p$ is at most $2k\ell d^\ell$. Hence, the inequality we want to show in order to apply Theorem 5 is

$$\frac{1}{C^k d^{2k}} \leq r_{2k} \cdot \prod_{\text{even } \ell=2}^{n} (1-r_\ell)^{2k\ell d^\ell}$$

First we note that

$$r_{2k} \cdot \prod_{\text{even } \ell=2}^{n} (1-r_\ell)^{2k\ell d^\ell} \geq r_{2k} \cdot \left( \prod_{\ell=2}^{\infty} (1-r_\ell)^{\ell d^\ell} \right)^{2k}$$

Since we will eventually choose $c = 4$, we know that $r_\ell \leq \frac{1}{2}$ for all $\ell \geq 2$. Hence $1 - r_\ell \geq \exp(-2r_\ell)$, and so,

$$(1-r_\ell)^{\ell d^\ell} \geq \exp(-2r_\ell \cdot \ell d^\ell) = \exp\left( \frac{-2\ell}{c^\ell} \right).$$

This gives us

$$r_{2k} \cdot \left( \prod_{\ell=2}^{\infty} (1-r_\ell)^{\ell d^\ell} \right)^{2k} \geq r_{2k} \cdot \exp\left( -2 \sum_{\ell=2}^{\infty} \frac{\ell}{c^\ell} \right)^{2k}.$$

We're in good shape, since the term in the exponential will end up being a small constant depending on $c$. Being precise, we have that $\sum_{\ell=0}^{\infty} \frac{\ell}{c^\ell} = \frac{c}{(c-1)^2}$, which follows by differentiating $\sum_{\ell=0}^{\infty} \frac{1}{c^\ell} = \frac{1}{1-\frac{1}{c}}$ and multiplying by $-c$. Hence, the above expression is

$$r_{2k} \cdot \exp\left( -\frac{2c}{(c-1)^2} \right)^{2k} = \frac{1}{(cd)^{2k}} \exp\left( -\frac{2c}{(c-1)^2} \right)^{2k}.$$

This means that it suffices to show that

$$\frac{1}{C^k d^{2k}} \leq \frac{1}{(cd)^{2k}} \exp\left( -\frac{2c}{(c-1)^2} \right)^{2k} \iff \frac{1}{C} \leq \frac{1}{c^2} \exp\left( -\frac{2c}{(c-1)^2} \right).$$

This is easy to achieve by picking the right parameters. For instance our choice $(c, C) = (4, 40)$ works.

3. **(0 pt.) [Tightness of the Lovasz Local Lemma]**

**This whole problem is optional and will not be graded.**

One version of the LLL that we saw asserts that for any set of events $A_1, \ldots, A_n$, such that for each $i$, $A_i$ is mutually independent of all but at most $d$ events, then as long as $\Pr[A_i] \leq \frac{1}{e(d+1)}$, then there is a nonzero chance of all events being simultaneously avoided.

(a) Define a set of events over a probability space such that each event is mutually independent of all but at most $d$ other events, and $\Pr[A_i] \leq 1/(d+1)$ for all $i$, but the probability of simultaneously avoiding all events $A_i$ is 0. This shows that the constant $e$ in the statement of the LLL cannot be replaced by 1.

(b) **(Challenge!)** For some constant $c \in (1, e)$, prove that the constant $e$ in the LLL cannot be replaced by $c$.

**SOLUTION:**

(a) We pick a uniformly random element $X$ from $\{1, 2, ..., d+1\}$. Let $A_i$ be the event that $X = i$. Clearly, $\Pr[A_i] = \frac{1}{d+1}$. Furthermore, there are $d+1$ total events, so each event can only depend on at most $d$ other events (and indeed they do). By design, we cannot simultaneously avoid all of the events $A_i$, since $X$ must be one of the elements $\{1, 2, ..., d+1\}$. Hence, this set of events satisfies the desired properties.

(b) See this paper: `https://page.mi.fu-berlin.de/szabo/PDF/k-s-SAT.pdf`.