Problem Set 8 CS265, Fall 2023
Due: Friday December 8 at 11:59PM PST
Group members: INSERT NAMES HERE

Please follow the homework policies on the course website.

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1. **(5 pt.) Sampling *Without* Replacement**

   Suppose there are n total balls, of which m are red. We sample k of the balls uniformly *without replacement.*[1] Let Z be the random variable denoting how many of the k balls are red. In this problem, you will show that Z is concentrated around its mean.

   (a) **(1 pt.)** Show that $\mathbb{E}[Z] = \frac{km}{n}$.

   (b) **(4 pt.)** When $k \geq 1$, show that $\Pr[|Z - \mathbb{E}[Z]| \geq \lambda] \leq 2e^{-\lambda^2/(2k)}$ for any $\lambda > 0$.
   [**HINT:** *Try applying the Azuma-Hoeffding tail bound to a Doob martingale. When applying Azuma-Hoeffding to a martingale* $\{Z_t\}$*, feel free to provide a short/intuitive explanation for why* $|Z_i - Z_{i-1}| \leq c_i$ *rather than a rigorous proof.*]

   (c) [**Optional: this won't be graded.**] When k is close to n, a tighter bound than that from part (b) holds.

      i. **(0 pt.)** When $k = n$, explain why $\Pr[Z = \mathbb{E}[Z]] = 1$.

      ii. **(0 pt.)** When $1 \leq k \leq n-1$, show that $\Pr[|Z - \mathbb{E}[Z]| \geq \lambda] \leq 2e^{-\lambda^2/(2v)}$ where v is defined as

      $$v := \sum_{i=1}^{k} \left(1 - \frac{k-i}{n-i}\right)^2.$$

      iii. **(0 pt.)** Show that $v \leq O(k(n-k)/n)$. This shows that the bound from part (c), ii is tighter than the bound from part (b) when k is close to n.

SOLUTION:

2. **(11 pt.) Reaching Consensus**

   This question considers a simple and fairly natural model of the dynamic of how opinions shift over time in a group.

   Suppose there is an undirected graph $G = (V, E)$ whose vertices represent the group members and a pair of members are friends if and only if they are connected by an edge. For simplicity, we assume that G contains none of the following: 1) self-loops, 2) multiple edges connecting the same pair of vertices, or 3) isolated vertices, i.e., vertices with no edge on them. Let S be the set of possible "opinions" on some topic, and lets suppose that each person has one and only one opinion on the topic at a time. (For concreteness, think of $S = \{A, B, C, ...\}$.) We can represent the opinions of the group members by a mapping $\sigma : V \to S$ where the group member corresponding to vertex v has opinion $\sigma(v)$.

[1] Note that this only makes sense when $k, m \leq n$.

The opinions σ of the group members evolve due to discussions between friends. We model the evolution of σ by the following time-homogeneous Markov chain: starting from the initial opinion σ_0, σ changes from σ_{t-1} to σ_t at step t as follows. Independently for every vertex v, we flip a fair coin. If the outcome is "heads", $\sigma_t(v)$ remains the same as $\sigma_{t-1}(v)$; otherwise, $\sigma_t(v)$ becomes $\sigma_{t-1}(v')$ for a uniformly random neighbor v' of v. In short, every group member keeps their own opinion with probability 1/2, and takes one of their friends' opinion with the remaining 1/2 probability.

In this problem, we will determine the likelihood that the group members reach a certain consensus, given their initial opinions.

(a) **(1 pt.)** If G is disconnected and $|S| > 1$, show that there exist initial opinions σ_0 of the members for which consensus is never reached.

(b) **(3 pt.)** If G is connected, show that consensus is eventually reached almost surely. That is, show that as the number of steps goes to infinity, the probability that consensus has been reached approaches 1.

(c) **(2 pt.)** Let X_t be the number of group members who have some opinion, say $A \in S$ after step t. Give an example where $(X_t)_{t\geq 0}$ is *not* a martingale with respect to $(\sigma_t)_{t\geq 0}$. The example should be one specific tuple (G, S, σ_0).

(d) **(3 pt.)** Let Y_t be the sum of the degrees of the vertices v corresponding to the group members with opinion A after step t. Prove that $(Y_t)_{t\geq 0}$ is a martingale with respect to $(\sigma_t)_{t\geq 0}$.

(e) **(2 pt.)** Assume that G is connected. What is the probability that all members of the group end up with opinion A (ie after some time, everyone has opinion, $A \in S$, for the rest of time)? Express your answer in terms of G and the initial opinion σ_0 of the group members.

[**HINT:** *Try applying the martingale stopping theorem to the martingale* $(Y_t)_{t\geq 0}$.]

SOLUTION: