Class 15: Agenda and Questions

1 Announcements

- HW7 out now (due after break)

- Tomorrow (Friday) is deadline to change to/from CR/NC, and also the withdrawal deadline.

- Have a nice fall break!

2 Questions?

Any questions from the minilectures and/or the quiz? (Coupling, Mixing times)

3 Shuffling I

In this exercise we'll practice coming up with a coupling.

> **Group Work**
>
> Consider the following Markov Chain for shuffling a deck of n cards:
>
> > At each timestep, choose a uniformly random card and move it to the top of the deck. (If you choose the top card, don't do anything).
>
> Let X_t denote the state of the deck after t steps.
>
> 1. Convince yourself that this Markov chain is irreducible and aperiodic. What is the stationary distribution of $X_0, X_1, \ldots$?
>
> 2. We are going to bound the mixing time of this Markov chain using *couplings*. Come up with a coupling on this Markov chain that you think will "couple" quickly.
>   **Hint**: *You might want to take inspiration from the graph-coloring example we saw, where we tried to make the same choice in both chains.*
>
> 3. For the coupling that you came up with, how long is it likely to take for the two chains to couple?
>   **Hint**: *Assuming you came up with the coupling that I think you did, it might be helpful to remember the* coupon collector's problem. *In particular, if you are trying to collect n coupons, the probability that you need more than $2n \log n$ tries to collect them all is $o(1)$.*

4. Come up with a bound on τ_{mix} using your coupling. Can you show that $\tau_{mix} = O(n \log n)$? (You may assume that n is sufficiently large).

4 Shuffling II

In this exercise, we'll see a different way to bound mixing times, other than coupling.

Group Work

Consider this **different shuffling scheme** for shuffling a deck of n cards:

At each timestep, choose the top card and move it to a uniformly random position in the deck. (Note that it is possible that we choose to keep the top card on the top, in which case nothing happens).

Let X_t denote the state of the deck after t steps.

1. Convince yourself that this chain is aperiodic and irreducible, and that the stationary distribution is uniform.

2. Let T be the first time at which the original bottom card of the deck is placed randomly somewhere. (That is, if the deck starts out with the ace of spades on the bottom, then time $T - 1$ is the first time that the ace of spades is on the top).

   Argue that, at any time $t \geq T$, the deck is completely uniform. (That is, the Markov chain has converged exactly to its stationary distribution).

3. What is $\mathbb{E}[T]$?

   Hint: *Assuming that the ace of spades is originally on the bottom, write*

   $T = $ *time it takes for the ace of spades to move to the second-from-bottom position*

   $\qquad + $ *time it takes for the ace of spades to move from the second-from-bottom position*

   $\qquad\qquad$ *to the third-from-bottom position*

   $\qquad + \cdots$

   and so on, and use linearity of expectation.

4. Notice that Markov's inequality implies that

$$\Pr[T \geq 2e\mathbb{E}[T]] \leq 1/(2e).$$

   Explain why this implies that the mixing time of $X_0, X_1, \ldots$ is at most $2e\mathbb{E}[T]$.

   Hint: *This is not quite as simple as saying "we just said it was fully mixed at time T," since the formal definition of the mixing time is a bit different.*

5   Strong Stationary Stopping Times

A random variable T is a **strong stationary stopping time** if:

- The event $T = t$ depends only on $X_1, \ldots, X_t$

- For all s, $\Pr[X_t = s | t \ge T] = \pi(s)$.

That is, you can tell if T has occurred based only on the steps so far, and once T occurs, the chain is completely mixed.

In the previous group work, we essentially showed that:

Theorem 1. *Let $X_0, X_1, \ldots$ be a Markov chain with stationary distribution π and let T be a strong stationary stopping time for this chain. Then*

$$\Delta(t) \le \Pr[T > t].$$

Group Work

(Bonus, if time.) Consider the following shuffling step for a deck of n cards:

- Assign each card a label "L" or "R," independently and uniformly at random.
- Put all the cards labeled "L" to the left, *preserving their relative order.* Put all the cards labeled "R" to the right, again preserving relative order.
- Put the "L" stack on top of the "R" stack.

You might recognize the as the *inverse* of a standard riffle shuffle. That is, if you do this process in reverse, you cut the deck at a random point and randomly interleave the two parts of the deck.

Use the method of strong stationary stopping times to show that the mixing time of this shuffle (repeated t times) is $O(\log n)$.