CS265, Fall 2023

Class 17: Agenda and Questions

1 Announcements

- HW7 due tomorrow.
- HW8 (last one!!!) out now.
- **You are all done with quizzes!**
- Final exam is Tuesday, December 12, 3:30-6:30pm.
- Practice exam released soon.
- Plan for Week 10:
  - Tuesday: Fun day on pseudorandomness (no quiz, not on HW or exam)
  - Thursday: The research frontier!

2 Questions?

Any questions from the minilectures and/or the quiz? (Stopping times, Martingale stopping theorem)

3 Wald's equation

In this exercise we'll get some practice applying the martingale stopping theorem, to prove **Wald's equation.**

Theorem 1 (Wald's equation)**.** *Suppose that $X_1, X_2, \ldots$ are non-negative i.i.d. random variables, distributed according to some random variable X. Let T be a stopping time for $\{X_i\}$. If $\mathbb{E}[X]$ and $\mathbb{E}[T]$ are both bounded, then*

$$\mathbb{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbb{E}[T] \cdot \mathbb{E}[X]. \tag{1}$$

Group Work

1. Wald's equation hopefully seems pretty intuitive. But there is something to prove! Come up with an example of some random variables X_i and T that don't obey the hypotheses of Theorem 1, so that the (1) does not hold.

Note: To make this more challenging, try to violate as few of the hypotheses as possible.

2. Let $Z_i = \sum_{j=1}^{i}(X_j - \mathbb{E}[X])$. Prove that $\{Z_i\}$ is a martingale with respect to $\{X_i\}$.
3. Argue that the martingale stopping theorem applies to $\{Z_i\}$ and T, where X, T are as in Theorem 1.
4. Use the Martingale stopping theorem to prove Wald's equation.
5. Consider rolling a fair, six-sided die repeatly. Let X be the sum of all of the rolls up until the first "6" is rolled, not including that 6. What is $\mathbb{E}X$?

4 Ballot Counting

Suppose that there is an election with two candidates, A and B, and n voters; say candidate A is the winner, receiving $N_A > N_B$ votes. (So $N_A + N_B = n$). The ballots are counted in a random order. What is the probably that A remained ahead for the entire count?

Let A_t be the number of votes for A at time t; let B_t be the number of votes for B at time t.

Let $Z_t = \frac{A_{n-t} - B_{n-t}}{n-t}$. That is, we imagine that we've already done the count, and then we "uncount" the votes one-by-one.

Group Work

1. Let T be the smallest t so that $Z_t = 0$; if this never occurs, set $T = n - 1$.

   Explain why T is a stopping time for $\{Z_t\}$, and why the Martingale Stopping Theorem applies to it. (Assume for now that $\{Z_t\}$ is indeed a martingale; you'll show that soon).
2. Apply the Martingale Stopping Theorem to $\{Z_t\}$ and T, and use it to compute the probability that candidate A was ahead throughout the count.
3. Show that $\{Z_t\}$ is a martingale. (Hint: It might help to think of the process that Z_t is tracking as follows. Start with two piles of ballots, one of size N_A and one of size N_B. Then choose a uniformly random vote to remove from one of the two piles; that will give you two piles corresponding to Z_1. Continue in this way.)