Class 5: Agenda and Questions

1 Warm-Up

Suppose you roll a 6-sided die n times. Use a Chernoff bound to bound the probability that you see more than $\frac{1+\delta}{6} \cdot n$ threes, where $\delta \in (0,1)$. What bound do you get as a function of n?

Group Work: Solutions

Let X be the number of threes that you see. Let X_i be an indicator random variable that is 1 iff you roll a three on roll i. Then $X = \sum_{i=1}^{n} X_i$, and $\mathbb{E}X_i = 1/6$. Thus, a Chernoff bound (for example, one of the simplified ones) says that

$$\Pr[X \geq (1+\delta) \cdot \frac{n}{6}] \leq \exp(-\mu\delta^2/3) = \exp(-n\delta^2/18) = \exp(-\Omega(n\delta^2)).$$

2 Announcements

- HW2 is due Friday!
- Friday is also the add-drop deadline; we'll get HW1 back to you before then.

3 Questions?

Any questions from the minilectures or warmup? (Moment generating functions; Chernoff bounds)

4 Randomized Routing

[Discussion with setup; the summary is below and also in more detail in the lecture notes.]

The goal is the following. Suppose we want to design a network with M nodes and a routing protocol in such a way that 1) we have relatively few edges in the network (ie $O(M)$ or $O(M \log M)$), and 2) if each node has a message to send to a some other node, the messages can all be routed to their destinations in a timely manner without too much congestion on the edges. More formally:

- Let H be the n-dimensional hypercube. There are 2^n vertices, each labeled with an element of $\{0,1\}^n$. Two vertices are connected by an edge if their labels differ in only one place. For example, 0101 is adjacent to 1101.
- Each vertex i has a packet (also named i), that it wants to route to another vertex $\pi(i)$, where $\pi : \{0,1\}^n \to \{0,1\}^n$ is a permutation.
- Each edge can only have one packet on it at a time (in each direction). Time is discrete (goes step-by-step), and the packets queue up in a first-in-first-out queue for each (directed) edge.

4.1 Group work: Bit-fixing scheme

Consider the following *bit-fixing scheme:* To send a packet i to a node j, we turn the bitstring i into the bitstring j by fixing each bit one-by-one, starting with the left-most and moving right. For example, to send

$$i = 001010$$

to

$$j = 101001,$$

we'd send

$$i = 001010 \to 101010 \to 101000 \to 101001 = j.$$

Group Work

1. Suppose that every packet is trying to get to $\vec{0}$ (the all-zero string). (Yes, I know that this isn't a permutation). Show that if every packet used the bit-fixing scheme (or, any scheme at all) to get to its destination, the total time required is at least $(2^n - 1)/n$ steps.

   Hint*: How many packets can arrive at $\vec{0}$ at any one timestep? How many packets need to arrive there?*

2. Suppose that n is even. Come up with an example of a permutation π where the bit-fixing scheme requires at least $(2^{n/2} - 1)/(n/2)$ steps.

   Hint*: Consider what happens if $(\vec{a}, \vec{b}) \in \{0,1\}^n$ wants to go to $(\vec{b}, \vec{a})$, where $\vec{a}, \vec{b} \in \{0,1\}^{n/2}$, and use part 1.*

3. If you still have time, think about the following: what happens if each packet i wants to go to a *uniformly random* destination $\delta(i)$, under the bit-fixing scheme? Will it be as bad as the scheme you came up with in part 2? Or will it finish in closer to $O(n)$ steps?

Group Work: Solutions

1. There are $2^n - 1$ packets that want to get to zero (not counting the packet that starts at zero, which is already there). At each timestep, at most n packets can go to zero, since there are only n edges coming out. Therefore we need at least $(2^n - 1)/n$ timesteps.
2. As in the hint, suppose that we construct a permutation π that sends $(\vec{a}, \vec{b})$ to $(\vec{b}, \vec{a})$. Then the bit-fixing scheme on $(\vec{a}, \vec{0})$ first proceeds by sending $(\vec{a}, \vec{0})$ to $\vec{0}$, for any $\vec{a}$. But there are $2^{n/2}$ choices for $\vec{a}$, and so by the previous part, this will take time at least $(2^{n/2} - 1)/(n/2)$.

4.2 A useful lemma

[Discussion explaining the following lemma.]

Lemma 1. *Let $D(i)$ denote the delay in the i'th packet. That is, this is the number of timesteps it spends waiting.*

Let $P(i)$ denote the path that packet i takes under the bit-fixing map. (So, $P(i)$ is a collection of directed edges).

Let $N(i)$ denote the number of other packets j so that $P(j) \cap P(i) \neq \emptyset$. That is, at some *point j wants to traverse an edge that i also wants to traverse, in the same direction, although possibly at some other point in time.*

Then $D(i) \leq N(i)$.

Group Work

Let $\delta : \{0,1\}^n \to \{0,1\}^n$ be a completely random function (not necessarily a permutation). That is, for each i, $\delta(i)$ is a uniformly random element of $\{0,1\}^n$, and each $\delta(i)$ is independent.

In this group work, you will analyze how the bit-fixing scheme performs when packet i wants to go to node $\delta(i)$.

Fix some special node/packet i. Let $D(i)$ and $P(i)$ be as above. Fix $\delta(i)$ (and hence $P(i)$, since we have committed to the bit-fixing scheme). But let's keep $\delta(j)$ random for all $j \neq i$. (Formally, we will condition on an outcome for $\delta(i)$; since $\delta(i)$ is independent from all of the other $\delta(j)$, this won't affect any of our calculations).

Let X_j be the indicator random variable that is 1 if $P(i)$ intersects $P(j)$.

1. Assume that we are using the bit-fixing scheme. Argue that $\mathbb{E}[\sum_j X_j] \leq n/2$.
   ***Hint**: In expectation, how many directed edges are in all of the paths $P(j)$ taken*

together (with repetition)? Show that this is at most $2^n \cdot n/2$. Then argue that the expected number of paths $P(j)$ that any single directed edge e is in is $1/2$. Finally, bound $\sum_j X_j \leq \sum_{e \in P(i)}$ (number of paths $P(j)$ that e is in) and use linearity of expectation and the fact that $|P(i)| \leq n$ to bound $\mathbb{E}[\sum_j X_j]$.

2. Use a Chernoff bound to bound the probability that $\sum_j X_j$ is larger than $10n$.

3. Use your answer to the previous question to bound the probability that the bit-fixing scheme takes more than $11n$ timesteps to send every packet i to $\delta(i)$, assuming that the destinations $\delta(i)$ are completely random.

   ***Hint**: Lemma 1.*

   If you still have time, think about the following:

4. However, the destinations are not random! They are given by some worst-case permutation π. Using what you've discovered above for random destinations, develop a randomized routing algorithm that gets every packet where it wants to go, with high probability, in at most $22n$ steps.

   ***Hint**: The fact that $22n$ is two times $11n$ is not an accident.*

Group Work: Solutions

1. The number of edges in all of the paths $P(j)$ is, in expectation,

$$\mathbb{E}[\sum_j \sum_e \mathbf{1}[e \in P(j)] = \sum_j \mathbb{E}[\text{length of path from } j \text{ to } \delta(j)] = \sum_j n/2 \leq 2^n \cdot n/2.$$

   This is because, for any j, the length of the bit-fixing path from j to $\delta(j)$ is just the number of coordinates on which j and $\delta(j)$ differ. But in expectation this is $n/2$, since the probability that they differ on any single coordinate is $1/2$. We also used the fact that there are $2^n - 1 \leq 2^n$ things in the sum.

   Thus, on average, every directed edge is in $1/2$ paths (since there are $n \cdot 2^n$ directed edges). By symmetry, the expected number of paths that any edge e must be in is $1/2$.

   Finally,

$$\mathbb{E}[\sum_j X_j] \leq \mathbb{E} \sum_{e \in P(i)} \sum_j \mathbf{1}[e \in P(j)],$$

   and by the above, $\mathbb{E} \sum_j \mathbf{1}[e \in P(j)]$ (which is the expected number of paths that e is in) is at most $1/2$. Thus,

$$\mathbb{E}[\sum_j X_j] \leq \sum_{e \in P(j)} \frac{1}{2} \leq \frac{n}{2}.$$

2. We have $\mathbb{E}[\sum_j X_j] \leq n/2 =: \mu$ by the previous part. By a Chernoff bound,

$$\Pr[\sum_j X_j \geq 10n] = \Pr[\sum_j X_j \geq 20\mu] \leq 2^{-20\mu} = 2^{-10n}.$$

3. The lemma says that the number of timesteps that packet i is delayed is at most the number of paths that cross $P(i)$, which is $\sum_j X_j$ using the notation from the previous problem. We just showed that this was at most $10n$ with probability 2^{-10n}. If this were to happen for all 2^n packets i, then the total time would be at most $11n$: at most n steps actually moving, and at most $10n$ steps delayed. We can union bound over all 2^n packets, to conclude that this indeed happens with probability at least $1 - 2^n 2^{-10n} = 1 - 2^{-9n}$.

4. Route to a random $\delta(i)$. Then route from $\delta(i)$ to $\pi(i)$. The total number of steps is at most $22n$ with high probability. Hooray!